An Application of HodgeRank to the Online Peer Assessment

Tse-Yu Lin* Yen-Lung Tsai†

Abstract

The biasedness and heterogeneity of peer assessment might cause an unfair score issue in the educational field. To deal with this problem, we propose a reference ranking method for online peer assessment system by HodgeRank. Such scheme provides instructors an objective scoring reference based on mathematics.

1 Introduction

In this paper, a ranking method based on HodgeRank is introduced to construct a reference score for online peer assessments. Peer assessments is a process for students to grade assignments of their peers' assignment [1, 5].

Peer assignment system is used to enhance students' future learning process, especially in higher education. Under such system, students are not only learning knowledge from textbooks and instructors, but also from the process to make judgement to peers' assignment. This procedure makes them to understand the weakness and strength from others, and then to review themselves.

However, there are some practical issues for peer assignment system. For example, students significantly give high grade than senior graders or professionals, see [2] for more detail. Also, students have a tendency to give grade within a range, the center of such range is often based on the first grade they gave. In this case, biasedness and heterogeneity occur in the peer assignment system.

In this paper, we propose a scheme to deal with this problem by HodgeRank, a statistical preference aggregation problem from pairwise comparison data. The purpose of HodgeRank is to d pairwise comparisons into a ranking. HodgeRank can not only generate a ranking order, but also provides inconsistency of the comparisons, see [4] for more detail.

These information provides a reference ranking order to instructors so that it becomes easier for instructors to give final score after all peer assignment process are finished.

*Department of Mathematical Sciences, National Chengchi University.

†Department of Mathematical Sciences, National Chengchi University.

2 HodgeRank

HodgeRank, a statistical ranking method based on combinatorial Hodge theory to find a consistent ranking. Rigorously speaking, HodgeRank is one solution of a graph Laplacian problem with minimum Euclidean norm.

Now, we start from notations borrowed from graph theory.

Consider a connected graph $\mathcal{G} = (V, E)$, where $V = \{1, 2, \cdots, n\}$ is the set of alternatives to be ranked, and $E \subseteq V \times V$, consists of some unordered pairs from V.

In this paper, V represents the set of students to be ranked by their peers, and E collects the information of pairwise comparisons. i.e., $(i, j) \in E$ if students i and j are compared at least once.

Denote Λ to be the number of assignments. Then for each assignment $\alpha \in \Lambda$, pairwise comparison data on a graph $\mathcal{G}$ of assignment α, is given by $Y^\alpha : E \to \mathbb{R}$ so that Y^α is skew-symmetry. i.e., $Y^\alpha_{ij} = -Y^\alpha_{ji}$ for all $i, j \in V$. $Y^\alpha_{ij} > 0$ if grade of the student j is higher than student i by Y^α_{ij} credits. For example, $Y^\alpha_{ij} \in [-100, 100]$ on hundred-mark system.

For each $\alpha \in \Lambda$, a weight matrix $W^\alpha = [w^\alpha_{ij}]$ is associated as follows: $w^\alpha_{ij} > 0$ if $Y^\alpha_{ij} \neq 0$, and 0 otherwise. Set $W = \sum_{\alpha\in\Lambda} W^\alpha$.

Let $Y = \sum_{\alpha\in\Lambda} Y^\alpha$ be a n-by-n matrix. The goal of the HodgeRank is find a ranking $s : V \to \mathbb{R}$ so that

$$Y_{ij} = s_j - s_i \text{ for all } i, j \in V. \tag{2.1}$$

However, equations (2.1) are, in general, not admissible. e.g., consider

$$Y = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

If there exists $s : V \to \mathbb{R}$ such that (2.1) hold. Then

$$1 = Y_{12} = s_2 - s_1 = (s_2 - s_3) + (s_3 - s_1) = Y_{32} + Y_{13} = 0$$

which leads to a contradiction. That is, it is impossible to solve (2.1) for any skew-symmetric matrix Y. Therefore, we should consider the least square solution of (2.1) instead. Before we rewrite above problem, we need to introduce some notations below.

Definition 2.1. [4] Denote

$$\mathcal{M}_G = \{X \in \mathbb{R}^{n\times n} \mid X_{ij} = s_i - s_j \text{for some } s : V \to \mathbb{R}\},$$

the space of global ranking, and the combinatorial gradient operator

$$\text{grad} : \mathcal{F}(V, \mathbb{R}) \to \mathcal{M}_G$$

is an operator defined from $\mathcal{F}(V, \mathbb{R})$, the set of all function from V to $\mathbb{R}$ (or the space of all potential functions), to $\mathcal{M}_G$, as follows

$$\big(\text{grad}s\big)(i, j) = s_j - s_i.$$

From the example above, it is easy to find that if $X = grad(s)$ for some $s \in \mathcal{F}(V, \mathbb{R})$, then $X_{ij} + X_{jk} + X_{ki} = 0$ for any $(i, j), (j, k), (k, i) \in E$. However, the converse might not be true in general. That is, denote

$$\mathcal{A} = \{X \in \mathbb{R}^{n\times n} \mid X^T = -X\},$$

the set of all skew-symmetric matrices, and let

$$\mathcal{M}_T = \{X \in \mathcal{A} \mid X_{ij} + X_{jk} + X_{ki} = 0\},$$

then $\mathcal{M}_G \subseteq \mathcal{M}_T$.

With these notations above, then the above problem becomes the following optimization problem:

$$\min_{X\in\mathcal{M}_G} ||X - Y||^2_{2,w} = \min_{X\in\mathcal{M}_G} \sum_{(i,j)\in E} w_{ij}(X_{ij} - Y_{ij})^2$$

That is, once a graph is given, then the weight on edge E determines an optimization problem. Conversely, a graph can intuitively arise from the ranking data.

Let $\{Y^\alpha \mid \alpha \in \Lambda\}$ be a set of n-by-n skew-symmetric matrices, and $\{W^\alpha \mid \alpha \in \Lambda\}$ is associated as above.

Then an undirected graph $\mathcal{G} = (V, E)$ can be defined by $V = \{1, 2, \cdots, n\}$ and

$$E = \{(i, j) \in V \times V \mid W_{ij} > 0\}.$$

In this case, we can treat X as a edge flow on $\mathcal{G}$ in the sense of combinatorial vector calculus.

In conclusion, we have the following relation between graph and

$$\mathcal{G} = (V, E) \Longleftrightarrow \begin{cases} X^T = -X \\ W = \sum\limits_{\alpha\in\Lambda} W^\alpha. \end{cases}$$

Hence, the optimization problem of a skew-symmetric least square problem can be view as an optimization problem of edge flow on a graph.

Definition 2.2. *(Consistency)* [4] Let $X : V \times X \to \mathbb{R}$ be a pairwise ranking edge flow on a graph $\mathcal{G} = (G, E)$.

- X is called consistency on $\{i, j, k\}$ if $(i, j), (j, k), (k, i) \in E$ and $X \in \mathcal{M}_T$
- X is called globally consistency on $\{i, j, k\}$ if $X = \text{grad}(s)$ for some $s \in \mathcal{F}(V, \mathbb{R})$

Note that if X is called globally consistency, then X is consistency on any 3-clique $\{i, j, k\}$, where $(i, j), (j, k), (k, i) \in E$.

Now, consider the weighted trace induced by W. i.e.,

$$< X, Y >= \text{tr}\big(X^T(W \odot Y)\big) = \sum_{(i,j)\in E} W_{ij}X_{ij}Y_{ij}$$

for $X, Y \in \mathcal{A}$, where $\odot$ represents the Hadamard product or elementwise product.

With this weighted inner product, we obtain two orthogonal complement of $\mathcal{A}$

$$\mathcal{A} = \mathcal{M}_G \oplus \mathcal{M}_G^\perp = \mathcal{M}_T \oplus \mathcal{M}_T^\perp$$

Since $\mathcal{M}_G \subseteq \mathcal{M}_T$, we have $\mathcal{M}_G^\perp \supseteq \mathcal{M}_T^\perp$ and we can get further orthogonal direct sum decomposition of $\mathcal{A}$ as follows:

$$\mathcal{A} = \mathcal{M}_G \oplus \mathcal{M}_H \oplus \mathcal{M}_T^\perp,$$

where $\mathcal{M}_H = \mathcal{M}_T \cap \mathcal{M}_G^\perp$.

This is called the combinatorial Hodge decomposition. For more detail about the theory of combinatorial Hodge decomposition, see [4].

One of the most useful result in [4] is the theorem below:

Theorem 2.1. [4]

1. The minimum norm solution s of (2.1) is equivalent to solve the following normal equation:

$$\Delta_0 s = -\text{div } Y,$$

where $\Delta_0 = \begin{cases} \sum\limits_{(i,j)} w_{ij} & \text{if } i = j \\ -w_{ij} & \text{if } j \in V \text{ such that } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$,

and

$\text{div}(Y)(i) = \sum\limits_{j s.t. (i,j)\in E} w_{ij}Y_{ij}$ is the combinatorial curl operator of Y.

2. The minimum norm solution s of (2.1) is

$$s^* = -\Delta_0^\dagger \text{div} Y.$$

Table 1: Number of components with respect to the number of assignments

Assignment #	1	2	3	4	5	6	7 ∼ 13
# of components	21	5	4	3	2	2	1

The Hodge decomposition indicates the solution of (2.1), while the theorem 2.1 shows how to calculate the minimum solution by solving the normal equation. In the next section, we display how to apply HodgeRank to the online peer assessment problem.

3 Application of HodgeRank to the online peer assessment problem.

As we mentioned at very beginning, the biasedness and heterogeneity lead to an unfair scoring result in the case of online peer assessment. Students usually grade other students' score based on the first score they gave. This causes the biasedness. However, since the score are usually compared with others. We can use this comparison behavior to reconstruct the true ranking.

The data we used in this section were collected from undergraduate calculus course. In this course, 133 students were asked to upload their GeogeBra [3] assignments. Then each student had to review five assignments randomly chosen from their peers to get partial credits.

There are 13 assignments during the semester, one key point of HodgeRank is based on the connectedness of the comparison graph.

From table 1 above, we can easily find that after half semester passed. The comparison matrix forms a connected graph so that we can apply HodgeRank to calculate the ranking of all students.

The traditional method to finalize peer assessment is either use average cumulative score or truncated average score, although these methods might have some statistical meaning, but they could not get rid of the biasedness and heterogeneity of peer assessment.

Figure 1 displays the results from cumulative score and from HodgeRank. We rescale score to $[0, 1]$ to compare their different.

There are two interesting situation from this figure. First, cumulative score offers a ranking higher than steady line. This reflects the biasedness and heterogeneity of traditional method. Second, ranking from HodgeRank gives a rather "normal" distributed curve so that the biasedness and heterogeneity can be eliminated by the normality.

In conclusion, this is the first time HodgeRank be applied in the field of education. Some numerical result are processed using real world data. However, there are some issue such as how to aggregate HodgeRank ranking method into peer assessment system still unsolved. This would be part of our future work to dig in.

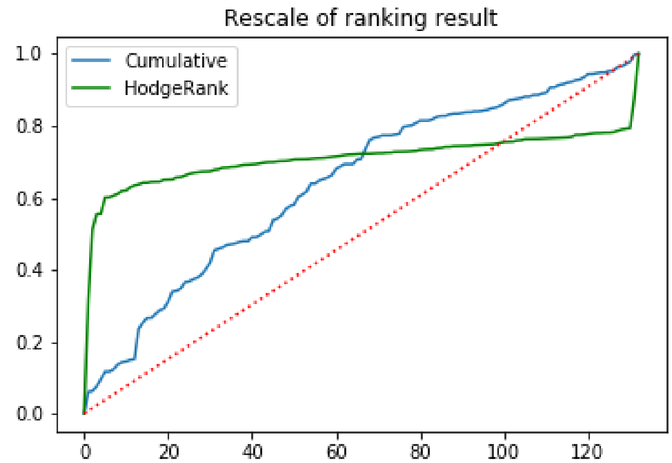

Figure 1: Results from HodgeRank and cumulative scoring

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